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Approximation of anisotropic classes by standard information☆

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Abstract

This paper deals with the approximation problem on anisotropic Besov classes $S^{\mathbf{r}}_{\mathbf{p}\theta}B(R^d)$, $\mathbf{p}=(p_1, \ldots, p_d)$, and Besov–Wiener classes $S^{\mathbf{r}}_{pq\theta}B(R^d)$ using standard information. The asymptotic decay rates of the best algorithms in the worst-case setting are determined.

Keywords: Information-based complexity; Optimal recovery; Anisotropic classes

1. Introduction and main results

Let R, R_+, Z, Z_+, and N be the sets of all real numbers, positive real numbers, integers, nonnegative integers, and positive integers, respectively. In addition, let Θ_σ be the set of all the sequences $\xi = \{\xi_v\}_{v \in Z^d}$ of points $\xi_v \in R^d$, $v \in Z^d$, satisfying the following conditions:

(i) $|\xi_{v_1}| \leqslant |\xi_{v_2}|$, if and only if $|v_1| \leqslant |v_2|$ for $v_1, v_2 \in Z^d$,
(ii) $\xi_{v_1} \neq \xi_{v_2}$, if and only if $v_1 \neq v_2$ for $v_1, v_2 \in Z^d$,
(iii)

$$\overline{\text{card}}\, \xi := \liminf_{a \to \infty} \frac{\text{card}(\xi \cap [-a, a]^d)}{(2a)^d} \leqslant \sigma.$$

Here $|\cdot|$ is the usual Euclidean norm, and card$E$, $E \subset R^d$, denotes the number of elements in the set E.

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Let $C(R^d)$ be the space of continuous functions defined on R^d and $X(R^d)$ a normed space of functions on R^d with the norm $\|\cdot\|_X$, $K \subset C(R^d) \cap X(R^d)$, the quantity

$$d(K, X) := \sup_{x(\cdot), y(\cdot) \in K} \|x(\cdot) - y(\cdot)\|_X$$

is called the diameter of *K*. For $\xi \in \Theta_\sigma$, the information of $f \in K$ is defined by $I_\xi f = \{f(\xi_v)\}_{v \in Z^d}$. I_ξ is called a standard sampling operator of the average cardinality $\leqslant \sigma$. The quantity

$$\Delta_\sigma(K, X) := \inf_{\xi \in \Theta_\sigma} \sup_{f \in K} d(I_\xi^{-1} I_\xi f \cap K, X)$$

is called the net width or the minimum information diameter of the set *K* in the space $X(R^d)$, where $I_\xi^{-1} I_\xi f$ denotes the set of all functions that share the same values as *f* on ξ. For any $\xi \in \Theta_\sigma$, a mapping $\varphi : I_\xi(K) \longrightarrow X(R^d)$ is called an algorithm, and $\varphi(I_\xi f)$ is called a recovering function (the approximation) of *f* in $X(R^d)$. Denote by Φ_ξ the set of all algorithms on *K*. If φ can be extended into a linear operator on the linearized set of *K*, we call the algorithm φ to be linear. Denote by Φ_ξ^{L} the set of all linear algorithms on the linearized set of *K*. The quantity

$$E_\sigma(K, X) := \inf_{\xi \in \Theta_\sigma} \inf_{\varphi \in \Phi_\xi} \sup_{f \in K} \|f - \varphi(I_\xi f)\|_X \tag{1.1}$$

is called the minimum intrinsic error of the optimal recovery of the set *K* in the space *X*. Taking Φ_ξ^{L} in the place of Φ_ξ in right side of (1.1), we denote by $E_\sigma^{\mathrm{L}}(K, X)$ the quantity obtained in this way, and call it the minimum linear intrinsic error. If *K* is a convex and centrally symmetric subset of *X*, then the following inequalities hold [14]:

$$\tfrac{1}{2}\Delta_\sigma(K, X) \leqslant E_\sigma(K, X) \leqslant E_\sigma^{\mathrm{L}}(K, X). \tag{1.2}$$

For $x = (x_1, \ldots, x_d) \in R_+^d$, $y = (y_1, \ldots, y_d) \in R_+^d$, we set $xy := (x_1 y_1, \ldots, x_d y_d)$, $1/x := (1/x_1, \ldots, 1/x_d)$, $x^\alpha := \prod_{i=1}^d x_i^\alpha$, $\alpha \in R$.

Let $\mathbf{p} = (p_1, \ldots, p_d)$, $1 \leqslant p_i \leqslant \infty$, $i = 1, 2, \ldots, d$, $I_j = (a_j, b_j)$, $-\infty \leqslant a_j < b_j \leqslant \infty$, $j = 1, \ldots, d$, $\mathbf{I} = I_1 \times \cdots \times I_d$. Denote by $L_{\mathbf{p}}(\mathbf{I})$ the Banach space of measurable functions $x(\cdot)$ on $\mathbf{I}$ with the vector norm (or mixed norm)

$$\|x(t)\|_{L_{\mathbf{p}}(\mathbf{I})} := \left(\int_{I_d} dt_d \left(\int_{I_{d-1}} dt_{d-1} \cdots \left(\int_{I_1} |x(t)|^{p_1} \, dt_1 \right)^{\frac{p_2}{p_1}} \cdots \right)^{\frac{p_d}{p_{d-1}}} \right)^{\frac{1}{p_d}}.$$

If $\mathbf{p} = (p, \ldots, p)$, then $L_{\mathbf{p}}(\mathbf{I})$ coincides with the usual space $L_p(\mathbf{I})$. For convenience, we write $\|\cdot\|_{\mathbf{p}}$ instead of $\|\cdot\|_{L_{\mathbf{p}}(R^d)}$, and denote $\mathbf{1} = (1, \ldots, 1)$, $\overline{\infty} = (\infty, \ldots, \infty)$. If $\mathbf{p} = (p_1, \ldots, p_d)$, $\mathbf{q} = (q_1, \ldots, q_d)$, $1 \leqslant p_i, q_i \leqslant \infty$, $i = 1, 2, \ldots, d$, then $\mathbf{p} \leqslant \mathbf{q}$ ($\mathbf{p} < \mathbf{q}$) means $p_i \leqslant q_i$ ($p_i < q_i$), $i = 1, \ldots, d$. Suppose that $\mathscr{J}_1, \ldots, \mathscr{J}_d$ are given finite sets. Let $\mathscr{J} = \mathscr{J}_1 \times \cdots \times \mathscr{J}_d$, $n = \operatorname{card} \mathscr{J}$, $a = \{a_j\}_{j \in \mathscr{J}}$, and

$$\|a\|_{l_{\mathbf{p}}^n(\mathscr{J})} := \left(\sum_{j_n} \left(\sum_{j_{n-1}} \cdots \left(\sum_{j_1} |a_{j_1, \ldots, j_d}|^{p_1} \right)^{\frac{p_2}{p_1}} \cdots \right)^{\frac{p_d}{p_{d-1}}} \right)^{\frac{1}{p_d}}$$

(see [1]). When $\mathscr{J} = \mathbf{Z}^d$, we write $\|\cdot\|_{l_{\mathbf{p}}}$ instead of $\|\cdot\|_{l_{\mathbf{p}}^\infty(Z^d)}$.

Let $1\leqslant q,\ p\leqslant\infty$. Denote by $L_{pq}(R^d)$ the normed linear space of functions defined on the Euclidean space R^d in which each function f is locally L_p-integrable and satisfies $\|f\|_{pq}<\infty$. Here the norm $\|\cdot\|_{pq}$ is defined by

$$\|f\|_{pq}:=\begin{cases}\left\{\sum\limits_{v\in Z^d}\|f(\cdot+v)\|^q_{L_p([0,1]^d)}\right\}^{\frac{1}{q}}, & 1\leqslant q<\infty,\\ \sup\limits_{v\in Z^d}\|f(\cdot+v)\|_{L_p([0,1]^d)}, & q=\infty,\end{cases}$$

where $\|\cdot\|_{L_p(D)}$ denotes the usual L_p-norm on the subset D of R^d, $L_{pq}(R^d)$ is a Banach space with the norm $\|\cdot\|_{pq}$. When $p=q$, $L_{qq}(R^d)=L_q(R^d)$ is the usual $L_q(R^d)$-space. When $d=1$, these notions may be seen in [4]. For convenience, we write $\|\cdot\|_p$ instead of $\|\cdot\|_{pp}$. If $p>q$, then the following relations:

$$\begin{aligned}&\|f\|_{pq}\geqslant\|f\|_p,\quad \|f\|_{pq}\geqslant\|f\|_q,\quad L_{pq}(R^d)\subset L_p(R^d)\cap L_q(R^d),\\ &\|f\|_{qp}\leqslant\|f\|_p,\quad \|f\|_{qp}\leqslant\|f\|_q,\quad L_{qp}(R^d)\supset L_p(R^d)\cup L_q(R^d)\end{aligned}\tag{1.3}$$

follow from the definitions.

Let $f(x)$, $x\in R^d$, be a measurable, almost everywhere finite real function. For $k_i\in N$, $t_i\in R$, denote by

$$\Delta^{k_i}_{t_i}f(x):=\sum_{l=0}^{k_i}(-1)^{l+k_i}\binom{k_i}{l}f(x_1,\ldots,x_i+lt_i,\ldots,x_d)$$

the k_ith difference of f at the point x for x_i with step t_i, $i=1,\ldots,d$.

Definition 1. Let $\mathbf{k}=(k_1,\ldots,k_d)\in N^d$, $\mathbf{r}=(r_1,\ldots,r_d)\in R^d_+$, $k_i>r_i$, $i=1,\ldots,d$, $1\leqslant\theta\leqslant\infty$, and $\mathbf{1}\leqslant\mathbf{p}<\overline{\infty}$. We say that $f\in B^{\mathbf{r}}_{\mathbf{p}\theta}(R^d)$ if the function f satisfies the conditions:

(i) $f\in L_{\mathbf{p}}(R^d)$,

(ii)

$$\|f\|_{b^{r_j}_{x_j\mathbf{p}\theta}(R^d)}:=\begin{cases}\left\{\int_R\left(\dfrac{\|\Delta^{k_j}_{t_j}f(\cdot)\|_{\mathbf{p}}}{|t_j|^{r_j}}\right)^{\theta}\dfrac{dt_j}{|t_j|}\right\}^{\frac{1}{\theta}}<\infty, & 1\leqslant\theta<\infty,\\ \sup\limits_{t_j\neq0}\dfrac{\|\Delta^{k_j}_{t_j}f(\cdot)\|_{\mathbf{p}}}{|t_j|^{r_j}}<\infty, & \theta=\infty\end{cases}$$

for each $j=1,\ldots,d$.

By [9], the linear space $B^{\mathbf{r}}_{\mathbf{p}\theta}(R^d)$ is a Banach space with the norm

$$\|f\|_{B^{\mathbf{r}}_{\mathbf{p}\theta}(R^d)}:=\|f\|_{\mathbf{p}}+\sum_{j=1}^d\|f\|_{b^{r_j}_{x_j\mathbf{p}\theta}(R^d)},$$

and it is called anisotropic Besov space. When $\theta = \infty$, $B^{\mathbf{r}}_{\mathbf{p}\theta}(R^d)$ coincides with the Hölder–Nikolski space $H^{\mathbf{r}}_{\mathbf{p}}(R^d)$. We define

$$S^{\mathbf{r}}_{\mathbf{p}\theta}B(R^d) := \{f \in L_{\mathbf{p}}(R^d) : \|f\|_{B^{\mathbf{r}}_{\mathbf{p}\theta}(R^d)} \leqslant 1\}.$$

Definition 2. Let $\mathbf{k} = (k_1, \ldots, k_d) \in Z^d_+$, $\mathbf{r} = (r_1, \ldots, r_d) \in R^d_+$, $k_i > r_i, i = 1, 2, \ldots, d$, $1 \leqslant \theta \leqslant \infty$, and $1 \leqslant q, p < \infty$. We say that $f \in B^{\mathbf{r}}_{pq\theta}(R^d)$ if the function f satisfies the following conditions:

(i) $f \in L_{pq}(R^d)$,
(ii)

$$\|f\|_{b^{r_i}_{x_i\, pq\theta}(R^d)} := \begin{cases} \left\{ \int_R \left(\dfrac{\|\Delta^{k_i}_{t_i} f(\cdot)\|_{pq}}{|t_i|^{r_i}} \right)^{\theta} \dfrac{dt_i}{|t_i|} \right\}^{\frac{1}{\theta}} < \infty, & 1 \leqslant \theta < \infty, \\ \sup\limits_{|t_i| \neq 0} \dfrac{\|\Delta^{k_i}_{t_i} f(\cdot)\|_{pq}}{|t_i|^{r_i}} < \infty, & \theta = \infty \end{cases}$$

for $i = 1, 2, \ldots, d$.

The linear space $B^{\mathbf{r}}_{pq\theta}(R^d)$ is a Banach space with the norm

$$\|f\|_{B^{\mathbf{r}}_{pq\theta}(R^d)} := \|f\|_{pq} + \sum_{i=1}^{d} \|f\|_{b^{r_i}_{x_i\, pq\theta}(R^d)},$$

and is called to be anisotropic Besov–Wiener space. When $p = q$, $B^{\mathbf{r}}_{pp\theta}(R^d) = B^{\mathbf{r}}_{p\theta}(R^d)$ is the usual anisotropic Besov space [9]. Set

$$S^{\mathbf{r}}_{pq\theta}B(R^d) := \{f \in B^{\mathbf{r}}_{pq\theta}(R^d) : \|f\|_{B^{\mathbf{r}}_{pq\theta}(R^d)} \leqslant 1\}.$$

Let $\mathbf{s} = (s_1, \ldots, s_d) \in Z^d_+$. For $\mathbf{1} \leqslant \mathbf{p} < \overline{\infty}$, denote by $L^{\mathbf{s}}_{\mathbf{p}}(R^d)$ the Sobolev spaces [9] consisting of all function $f \in L_{\mathbf{p}}(R^d)$ with the finite norm

$$\|f\|_{L^{\mathbf{s}}_{\mathbf{p}}(R^d)} := \|f\|_{\mathbf{p}} + \sum_{i=1}^{d} \left\| \frac{\partial^{s_i} f}{\partial x_i^{s_i}} \right\|_{\mathbf{p}}.$$

For $1 \leqslant q, p < \infty$, denote by $L^{\mathbf{s}}_{qp}(R^d)$ the Sobolev–Wiener spaces consisting of all function $f \in L_{qp}(R^d)$ with the finite norm

$$\|f\|_{L^{\mathbf{s}}_{qp}(R^d)} := \|f\|_{qp} + \sum_{i=1}^{d} \left\| \frac{\partial^{s_i} f}{\partial x_i^{s_i}} \right\|_{qp}.$$

An extensive literature is devoted to the question of recovering functions from their values at nodes and closely related concepts of widths (see the survey of [8,10,11,14]). Temlyakov [12,13] considered the optimal recovery of periodic functions on the class of functions with bounded mixed derivative. Sun Yongsheng [11] considered an optimal recovery problem concerning a class of differentiable functions defined on the whole real axis. Wang and Sun [16] discussed the Kolmogorov widths between the anisotropic space and the space of functions with mixed smoothness. The authors of [17] studied the widths of Besov classes in the usual Sobolev spaces. In [3] some problems of the optimal recovery of periodic Besov classes were studied. In this paper, we continue our investigation [5,6] and obtain the following results.

Theorem 1. *Let* $\mathbf{k} = (k_1, \ldots, k_d) \in N^d, \mathbf{r} = (r_1, \ldots, r_d) \in R^d_+, \mathbf{s} = (s_1, \ldots, s_d) \in Z^d_+$, $k_j > r_j > s_j, j = 1, \ldots, d, \omega = 1 - \sum_{i=1}^d \frac{1}{r_i}(\frac{1}{p_i} - \frac{1}{q_i}) > 0, \sum_{i=1}^d \frac{1}{r_i\omega} < 1, 1 \leqslant \theta \leqslant \infty, \mathbf{1} < \mathbf{p} \leqslant \mathbf{q} < \overline{\infty}, q_1 \leqslant q_2 \leqslant \cdots \leqslant q_d$, *and* $\sigma \geqslant 1$. *Then*

$$\begin{aligned}\sigma^{-a(\omega-\eta)} &\ll \tfrac{1}{2}\Delta_\sigma(S^{\mathbf{r}}_{\mathbf{p}\theta}B(R^d), L^{\mathbf{s}}_{\mathbf{q}}(R^d)) \leqslant E_\sigma(S^{\mathbf{r}}_{\mathbf{p}\theta}B(R^d), L^{\mathbf{s}}_{\mathbf{q}}(R^d))\\ &\leqslant E^{\mathrm{L}}_\sigma(S^{\mathbf{r}}_{\mathbf{p}\theta}B(R^d), L^{\mathbf{s}}_{\mathbf{q}}(R^d)) \ll \sigma^{-a(\omega-\eta)},\end{aligned}$$

where $a = (\sum_{j=1}^d \frac{1}{r_j})^{-1},\ \eta = \max\limits_{1\leqslant i\leqslant d} \frac{s_i}{r_i}$.

Theorem 2. *Make the same assumptions on* $\mathbf{k}, \mathbf{r}, \mathbf{s}, \theta, \sigma, a$ *and* η *as in Theorem* 1. *If* $\sum_{j=1}^d \frac{1}{r_j} < 1$, *and* $1 < q \leqslant p < \infty$. *Then*

$$\begin{aligned}\sigma^{-a(1-\eta)} &\ll \tfrac{1}{2}\Delta_\sigma(S^{\mathbf{r}}_{pq\theta}B(R^d), L^{\mathbf{s}}_{q}(R^d)) \leqslant E_\sigma(S^{\mathbf{r}}_{pq\theta}B(R^d), L^{\mathbf{s}}_{q}(R^d))\\ &\leqslant E^{\mathrm{L}}_\sigma(S^{\mathbf{r}}_{pq\theta}B(R^d), L^{\mathbf{s}}_{q}(R^d)) \ll \sigma^{-a(1-\eta)}.\end{aligned}$$

Theorem 3. *Make the same assumptions on* $\mathbf{k}, \mathbf{r}, \mathbf{s}, \theta, \sigma, a$ *and* η *as in Theorem* 2. *Then*

$$\begin{aligned}\sigma^{-a(1-\eta)} &\ll \tfrac{1}{2}\Delta_\sigma(S^{\mathbf{r}}_{p\theta}B(R^d), L^{\mathbf{s}}_{qp}(R^d)) \leqslant E_\sigma(S^{\mathbf{r}}_{p\theta}B(R^d), L^{\mathbf{s}}_{qp}(R^d))\\ &\leqslant E^{\mathrm{L}}_\sigma(S^{\mathbf{r}}_{p\theta}B(R^d), L^{\mathbf{s}}_{qp}(R^d)) \ll \sigma^{-a(1-\eta)}.\end{aligned}$$

In the sequel, $c, c_1, c'_1, \ldots$ denote constants which depend only on d, $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ and $\mathbf{s}$.

2. Some lemmas

Let $v = (v_1, \ldots, v_d) \in R^d_+$, $\mathbf{p} = (p_1, \ldots, p_d)$, $\mathbf{1} \leqslant \mathbf{p} \leqslant \overline{\infty}$. We shall denote by $E_{v\mathbf{p}}(R^d)$ the restriction to R^d of the space of all entire functions of exponential type v which belong to $L_{\mathbf{p}}(R^d)$.

We shall use the following auxiliary lemmas.

Lemma 1. *Let* $v = (v_1, \ldots, v_d) \in R^d_+$, $f \in E_{v\mathbf{p}}(R^d)$, $\mathbf{1} < \mathbf{p} < \overline{\infty}$. *Then*

(a) $f(x) = \sum_{k\in Z^d} f\left(\frac{k\pi}{v}\right) \sin c_d(v(x - k\pi/v))$, *the series on the right-hand side is uniformly absolutely convergent for all* $x \in R^d$, *where* $\sin c_d(x) = \prod_{j=1}^d \sin c(x_j)$, $\sin c(u) = u^{-1}\sin u$ *for* $u \neq 0$, *and* 1 *for* $u = 0$.
(b) $\|f - \sum_{|k_l|\leqslant m_l} f(k\pi/v) \sin c_d(v(x - k\pi/v))\|_{L_{\mathbf{p}}(R^d)} \to 0,\ m_l \to \infty, l = 1, \ldots, d$.
(c) *There exists a constant* $c_{\mathbf{p}}$ *such that* $\|f\|_{L_{\mathbf{p}}(R^d)} \leqslant c_{\mathbf{p}} \prod_{i=1}^d (\frac{\pi}{v_i})^{1/p_i} \|f(k\pi/v)\|_{l_{\mathbf{p}}(Z^d)}$.

The case for $\mathbf{p} = (p, \ldots, p)$, Lemma 1 is given in [2,15], and in more general form in [5].

Lemma 2 (*Nikol'skii [9], Jiang [5], Magaril-Il'jaev [7]*). *Suppose that* $\mathbf{l}=(l_1,\ldots,l_d)\in Z_+^d$, $\boldsymbol{v}\in R_+^d$, $\boldsymbol{v}>\mathbf{1}$, $\mathbf{1}<\mathbf{p}\leqslant\mathbf{q}<\overline{\infty}$, *then there exists a constant* $c=c(\mathbf{l},\mathbf{p},\mathbf{q})$ *such that for any* $f\in E_{v\mathbf{p}}(R^d)$,

$$\|f^{(\mathbf{l})}\|_{\mathbf{q}}\leqslant c\boldsymbol{v}^{\mathbf{l}+1/\mathbf{p}-1/\mathbf{q}}\|f\|_{\mathbf{p}}=c\prod_{i=1}^{d}v_i^{l_i+1/p_i-1/q_i}\|f\|_{\mathbf{p}}.$$

Lemma 3. *Let* $\mathbf{1}<\mathbf{p}<\overline{\infty}$, $\mathbf{s}=(s_1,\ldots,s_d)\in Z_+^d$, *and* $\boldsymbol{v}=(v_1,\ldots,v_d)\in R_+^d$. *Then for every* $f\in L_{\mathbf{p}}(R^d)\cap C(R^d)$,$\{f(k\pi/v)\}_{k\in Z^d}\in l_{\mathbf{p}}(Z^d)$, $\frac{\partial^{s_i}f}{\partial x_i^{s_i}}\in L_{\mathbf{p}}(R^d)$, $i=1,2,\ldots,d$, *there exists a unique interpolating function* $L_v(f,\cdot)\in E_{v\mathbf{p}}(R^d)$ *such that* $L_v(f,k\pi/v)=f(k\pi/v)$, $k\in Z^d$. *In addition, for any* $g\in E_{v\mathbf{p}}(R^d)$

(i) $\|f-L_v(f)\|_{\mathbf{p}}\leqslant c_{\mathbf{p}}\prod_{i=1}^d(\frac{\pi}{v_i})^{\frac{1}{p_i}}\|f(k\pi/v)-g(k\pi/v)\|_{l_{\mathbf{p}}(Z^d)}+\|f-g\|_{L_{\mathbf{p}}(R^d)}$.

(ii) $\|\frac{\partial^{s_i}f}{\partial x_i^{s_i}}-\frac{\partial^{s_i}L_v(f)}{\partial x_i^{s_i}}\|_{\mathbf{p}}\leqslant c_{\mathbf{p}}v_i^{s_i}\prod_{i=1}^d(\frac{\pi}{v_i})^{\frac{1}{p_i}}\|f(k\pi/v)-g(k\pi/v)\|_{l_{\mathbf{p}}(Z^d)}+\|\frac{\partial^{s_i}f}{\partial x_i^{s_i}}-\frac{\partial^{s_i}g}{\partial x_i^{s_i}}\|_{L_{\mathbf{p}}(R^d)}$, $i=1,2,\ldots,d$.

The case for $\mathbf{p}=(p,\ldots,p)$, (i) is given in [2,15], and in more general form in [5]. By (i) and Lemma 2, we can get (ii).

Lemma 4. *Let* $\mathbf{1}<\mathbf{p}<\overline{\infty}$, $p_1\leqslant p_2\leqslant\cdots\leqslant p_d$, *and* $f\in L_{\mathbf{p}}(R^d)$. *Suppose that for every* $s=(s_1,\ldots,s_d)\in Z_+^d$ *with* $|s|:=s_1+s_2+\cdots+s_d\leqslant d$, *all generalized derivatives* $\frac{\partial^{|s|}f}{\partial x_1^{s_1}\cdots\partial x_d^{s_d}}$ *of f exit and* $\frac{\partial^{|s|}f}{\partial x_1^{s_1}\cdots\partial x_d^{s_d}}\in L_{\mathbf{p}}(R^d)$. *Then*

$$\prod_{i=1}^{d}\left(\frac{\pi}{v_i}\right)^{\frac{1}{p_i}}\|f(k\pi/v)\|_{l_{\mathbf{p}}(Z^d)}\leqslant\|f\|_{\mathbf{p}}+\sum_{1\leqslant i\leqslant d}\frac{\pi}{v_i}\|f'_{x_i}\|_{\mathbf{p}}+\sum_{1\leqslant i<j\leqslant d}\frac{\pi}{v_i}\cdot\frac{\pi}{v_j}\left\|\frac{\partial^2 f}{\partial x_i\partial x_j}\right\|_{\mathbf{p}}$$
$$+\cdots+(\pi/v)^{1}\left\|\frac{\partial^d f}{\partial x_1\cdots\partial x_d}\right\|_{\mathbf{p}}.$$

Proof. For $p=(p,\ldots,p)$ the lemma is proved in [2,15]. The argument in the general case is similar. But for the sake of readability, we give the proof in some detail.

First consider the case $d=2$. The argument in higher dimensions is analogous. Let $x_i=\frac{i\pi}{v_1}$, $y_j=\frac{j\pi}{v_2}$, $i,j\in Z$. Then

$$|f(x_i,y_j)|\leqslant\frac{v_2}{\pi}\int_{y_j}^{y_{j+1}}|f(x_i,y)|\,dy+\int_{y_j}^{y_{j+1}}\left|\frac{\partial}{\partial y}f(x_i,y)\right|dy,$$

$$|f(x_i,y)|\leqslant\frac{v_1}{\pi}\int_{x_i}^{x_{i+1}}|f(x,y)|\,dx+\int_{x_i}^{x_{i+1}}\left|\frac{\partial}{\partial x}f(x,y)\right|dx,$$

$$\left|\frac{\partial}{\partial y}f(x_i,y)\right|\leqslant\frac{v_1}{\pi}\int_{x_i}^{x_{i+1}}\left|\frac{\partial}{\partial y}f(x,y)\right|dx+\int_{x_i}^{x_{i+1}}\left|\frac{\partial^2}{\partial x\partial y}f(x,y)\right|dx.$$

Hence

$$
\begin{aligned}
|f(x_i, y_j)| \leqslant{} & \frac{v_1}{\pi}\frac{v_2}{\pi}\int_{y_j}^{y_{j+1}}\int_{x_i}^{x_{i+1}} |f(x,y)|\,dx\,dy + \frac{v_2}{\pi}\int_{y_j}^{y_{j+1}}\int_{x_i}^{x_{i+1}} \left|\frac{\partial}{\partial x} f(x,y)\right| dx\,dy \\
& + \frac{v_1}{\pi}\int_{y_j}^{y_{j+1}}\int_{x_i}^{x_{i+1}} \left|\frac{\partial}{\partial y} f(x,y)\right| dx\,dy \\
& + \int_{y_j}^{y_{j+1}}\int_{x_i}^{x_{i+1}} \left|\frac{\partial^2}{\partial x \partial y} f(x,y)\right| dx\,dy.
\end{aligned}
\tag{2.1}
$$

For a function of two variables, $g(x, y)$, by the Hölder integral inequality, we have

$$
\begin{aligned}
& \sum_{i\in Z}\left(\int_{y_j}^{y_{j+1}}\int_{x_i}^{x_{i+1}} |g(x,y)|\,dx\,dy\right)^{p_1} \\
& \quad \leqslant \sum_{i\in Z}\left(\frac{\pi}{v_1}\cdot\frac{\pi}{v_2}\right)^{\frac{p_1}{q_1}} \int_{y_j}^{y_{j+1}}\int_{x_i}^{x_{i+1}} |g(x,y)|^{p_1}\,dx\,dy \\
& \quad = \left(\frac{\pi}{v_1}\cdot\frac{\pi}{v_2}\right)^{\frac{p_1}{q_1}} \int_{y_j}^{y_{j+1}}\left(\int_R |g(x,y)|^{p_1}\,dx\right)dy, \\
& \sum_{j\in Z}\left(\sum_{i\in Z}\left(\int_{y_j}^{y_{j+1}}\int_{x_i}^{x_{i+1}} |g(x,y)|\,dx\,dy\right)^{p_1}\right)^{\frac{p_2}{p_1}} \\
& \quad \leqslant \sum_{j\in Z}\left(\left(\frac{\pi}{v_1}\cdot\frac{\pi}{v_2}\right)^{\frac{p_1}{q_1}} \int_{y_j}^{y_{j+1}}\left(\int_R |g(x,y)|^{p_1}\,dx\right)dy\right)^{\frac{p_2}{p_1}} \\
& \quad \overset{\frac{p_2}{p_1}\geqslant 1}{\leqslant} \left(\frac{\pi}{v_1}\cdot\frac{\pi}{v_2}\right)^{\frac{p_2}{q_1}} \sum_{j\in Z}\left(\left(\frac{\pi}{v_2}\right)^{\frac{p_2-p_1}{p_2}}\left(\int_{y_j}^{y_{j+1}}\left(\int_R |g(x,y)|^{p_1}\,dx\right)^{\frac{p_2}{p_1}} dy\right)^{\frac{p_1}{p_2}}\right)^{\frac{p_2}{p_1}} \\
& \quad = \left(\frac{\pi}{v_1}\right)^{\frac{p_2}{q_1}}\left(\frac{\pi}{v_2}\right)^{\frac{p_2}{q_1}+\frac{p_2-p_1}{p_1}} \int_R\left(\int_R |g(x,y)|^{p_1}\,dx\right)^{\frac{p_2}{p_1}} dy,
\end{aligned}
\tag{2.2}
$$

where $\frac{1}{q_1}+\frac{1}{p_1}=1$. By (2.1) and (2.2), and the Minkowski integral inequality, we complete the proof of Lemma 4. □

Lemma 5 (*Nikol'skii [9]*). *Suppose that* $\mathbf{l}=(l_1,\ldots,l_d)\in Z_+^d$, $\mathbf{r}=(r_1,\ldots,r_d)\in R_+^d$, $\mathbf{1}\leqslant\mathbf{p}\leqslant\overline{\infty}$, $1\leqslant\theta\leqslant\infty$, $\sum_{i=1}^d \frac{l_i}{r_i}<1$, *and* $f\in B^{\mathbf{r}}_{\mathbf{p}\theta}(R^d)$. *Then*

$$
\|f^{(\mathbf{l})}\|_{B^{\mathbf{u}}_{\mathbf{p}\theta}(R^d)} \leqslant c\|f\|_{B^{\mathbf{r}}_{\mathbf{p}\theta}(R^d)},
$$

where $\mathbf{u}=(1-\sum_{k=1}^d \frac{l_k}{r_k})\mathbf{r}$.

Lemma 6 (*Nikol'skii [9] and Jiang [5]*). *Let* $\mathbf{1} < \mathbf{p} \leqslant \mathbf{q} < \overline{\infty}$, $1 \leqslant \theta \leqslant \infty$, $\omega = 1 - \sum_{i=1}^{d} \frac{1}{r_i}(\frac{1}{p_i} - \frac{1}{q_i}) > 0$, *and* $\mathbf{r}' = \omega \mathbf{r}$. *If* $f \in B^{\mathbf{r}}_{\mathbf{p}\theta}(R^d)$, *then*

$$\|f\|_{B^{\mathbf{r}'}_{\mathbf{q}\theta}(R^d)} \leqslant c \|f\|_{B^{\mathbf{r}}_{\mathbf{p}\theta}(R^d)}.$$

3. Proof of Theorem 1

Proof of Theorem 1. To find the upper bound, for any real number $\rho > 0$, we define

$$g_\rho(t) := \lambda_{\rho,s}^{-1} \left(\frac{\sin \rho t}{t} \right)^{2s} \qquad (t \in R, 2s > 1),$$

$$\lambda_{\rho,s} := \int_R \left(\frac{\sin \rho t}{t} \right)^{2s} dt \asymp \rho^{2s-1}, \quad \rho \to \infty,$$

while the notation "$f(\rho) \asymp g(\rho)$" means that there exist constants $c_1 > 0$ and $c_2 > 0$ $(c_1 < c_2)$ such that $c_1|g(\rho)| \leqslant |f(\rho)| \leqslant c_2|g(\rho)|$ for every sufficiently large ρ. If $\alpha > 0$ and $2s > 1 + \alpha$, then

$$\int_R g_\rho(t)|t|^\alpha \, dt \asymp \rho^{-\alpha}, \quad \rho \to \infty.$$

Let $\rho_i > 0$, $i = 1, \ldots, d$. For every $f \in B^{\mathbf{r}}_{\mathbf{p}\theta}(R^d)$, we have

$$\begin{aligned} T_{\rho_i}(f, x) &:= \int_R g_{\rho_i}(t_i)((-1)^{k_i+1}\Delta_{t_i}^{k_i} f(x) + f(x)) \, dt_i \\ &= \int_R g_{\rho_i}(t_i) \sum_{j=1}^{k_i} d_j f(x_1, \ldots, x_{i-1}, x_i + jt_i, x_{i+1}, \ldots, x_d) \, dt_i \\ &= \int_R G_{\rho_i}(t_i - x_i) f(x_1, \ldots, x_{i-1}, t_i, x_{i+1}, \ldots, x_d) \, dt_i, \end{aligned}$$

where $\sum_{j=1}^{k_i} d_j = 1$ and $G_{\rho_i}(t) = \sum_{j=1}^{k_i} \frac{d_j}{j} g_{\rho_i}(\frac{t}{j})$, $i = 1, 2, \ldots, d$. Set

$$T_{\rho_1,\ldots,\rho_n}(f, x) := \int_{R^n} G_{\rho_1}(u_1) \cdots G_{\rho_n}(u_n) f(x_1 + u_1, \ldots, x_n + u_n, x_{n+1}, \ldots, x_d) \, du,$$

$1 \leqslant n \leqslant d$. Then for $\mathbf{1} \leqslant \mathbf{p} \leqslant \mathbf{q} < \overline{\infty}$, according to a result from [9], $T_{\rho_1,\ldots,\rho_d} \in E^{\mathbf{p}}_{\nu}(R^d) \subset E^{\mathbf{q}}_{\nu}(R^d)$, with $\nu = (2s\rho_1, \ldots, 2s\rho_d)$. Let $2s > 1 + \max\{r_i, i = 1, \ldots, d\}$, $2s\rho_j = \nu_j = \pi\sigma^{\frac{a}{r_j}}$, $j = 1, \ldots, d$. By [5],

$$\|f(\cdot) - T_{\rho_1,\ldots,\rho_d}(f, \cdot)\|_{\mathbf{q}} \leqslant c\sigma^{-a\omega} \|f\|_{B^{\mathbf{r}}_{\mathbf{p}\theta}(R^d)},$$

$$\prod_{i=1}^{d} \left(\frac{\pi}{\nu_i} \right)^{\frac{1}{p_i}} \|f(k\pi/\nu) - T_{\rho_1,\ldots,\rho_d}(k\pi/\nu)\|_{\mathbf{q}} \leqslant c\sigma^{-a\omega} \|f\|_{B^{\mathbf{r}}_{\mathbf{p}\theta}(R^d)}. \tag{3.1}$$

Let

$$\mathbf{r}' = (r_1', \ldots, r_d'), \quad r_j' = r_j \left(1 - \sum_{i=1}^{d} \frac{1}{r_i} \left(\frac{1}{p_i} - \frac{1}{q_i} \right) \right) = r_j\omega, \quad j = 1, \ldots, d.$$

By the Minkowski and Hölder integral inequality, for $1 \leqslant i \leqslant d$, we have

$$\begin{aligned}\left\| \frac{\partial^{s_i}}{\partial x_i^{s_i}} f(x) - \frac{\partial^{s_i}}{\partial x_i^{s_i}} T_{\rho_1}(f, x)] \right\|_{\mathbf{q}} &= \left\| \int_R g_{\rho_1}(t_1) \Delta_{t_1}^{k_1} \frac{\partial^{s_i}}{\partial x_i^{s_i}} f(x)\, dt_1 \right\|_{\mathbf{q}} \\ &\leqslant \int_R \left\| \Delta_{t_1}^{k_1} \frac{\partial^{s_i}}{\partial x_i^{s_i}} f(\cdot) \right\|_{\mathbf{q}} g_{\rho_1}(t_1)\, dt_1 \\ &\leqslant \left(\int_R \left(\frac{\left\| \Delta_{t_1}^{k_1} \frac{\partial^{s_i}}{\partial x_i^{s_i}} f(\cdot) \right\|_{\mathbf{q}}}{|t_1|^{r_1'(1-\frac{s_i}{r_i'})+\frac{1}{\theta}}} \right)^{\theta} dt_1 \right)^{\frac{1}{\theta}} \\ &\quad \times \left(\int_R |t_1|^{(r_1'(1-\frac{s_i}{r_i'})+\frac{1}{\theta})\theta'} |g_{\rho_1}(t_1)|^{\theta'}\, dt_1 \right)^{\frac{1}{\theta'}} \\ &\leqslant c_1 \rho_1^{-r_1'(1-\frac{s_i}{r_i'})} \left\| \frac{\partial^{s_i}}{\partial x_i^{s_i}} f(\cdot) \right\|_{b_{x_1 \mathbf{q} \theta}^{r_1'(1-\frac{s_i}{r_i'})}(R^d)}, \end{aligned} \tag{3.2}$$

where $\theta + \theta' = 1$. Moreover, we have

$$\begin{aligned} &\left\| \frac{\partial^{s_i}}{\partial x_i^{s_i}} T_{\rho_1}(f, x) - \frac{\partial^{s_i}}{\partial x_i^{s_i}} T_{\rho_1,\rho_2}(f, x) \right\|_{\mathbf{q}} \\ &\quad = \left\| \int_R G_{\rho_1}(t_1) \frac{\partial^{s_i}}{\partial x_i^{s_i}} h_1(x_1 + t_1, x_2, \ldots, x_d)\, dt_1 \right\|_{\mathbf{q}} \\ &\quad \leqslant \int_R g_{\rho_1}(t_1) \left\| \frac{\partial^{s_i}}{\partial x_i^{s_i}} h(\cdot) \right\|_{\mathbf{q}} dt_1 = \left\| \frac{\partial^{s_i}}{\partial x_i^{s_i}} h(\cdot) \right\|_{\mathbf{q}}, \end{aligned} \tag{3.3}$$

where

$$h(x_1, x_2, \ldots, x_d) = f(x_1, x_2, \ldots, x_d) - \int_R G_{\rho_2}(t_2) f(x_1, x_2 + t_2, x_3, \ldots, x_d)\, dt_2.$$

Similarly to (3.2), we can get

$$\left\| \frac{\partial^{s_i}}{\partial x_i^{s_i}} h(\cdot) \right\|_{\mathbf{q}} \leqslant c_2 \rho_2^{-r_2'(1-\frac{s_i}{r_i'})} \left\| \frac{\partial^{s_i}}{\partial x_i^{s_i}} f(\cdot) \right\|_{b_{x_2 \mathbf{q} \theta}^{r_2'(1-\frac{s_i}{r_i'})}(R^d)}. \tag{3.4}$$

In the case $2 \leqslant j \leqslant d$, a proper calculation yields

$$\begin{aligned} &\left\| \frac{\partial^{s_i}}{\partial x_i^{s_i}} T_{\rho_1,\ldots,\rho_{j-1}}(f, x) - \frac{\partial^{s_i}}{\partial x_i^{s_i}} T_{\rho_1,\ldots,\rho_j}(f, x) \right\|_{\mathbf{q}} \\ &\quad \leqslant c_j \rho_j^{-r_j'(1-\frac{s_i}{r_i'})} \left\| \frac{\partial^{s_i}}{\partial x_i^{s_i}} f \right\|_{b_{x_j \mathbf{q} \theta}^{r_j'(1-\frac{s_i}{r_i'})}(R^d)}. \end{aligned} \tag{3.5}$$

Hence, by (3.2)–(3.5), Lemmas 5 and 6, we have

$$\begin{aligned}
&\left\| \frac{\partial^{s_i}}{\partial x_i^{s_i}} f(x) - \frac{\partial^{s_i}}{\partial x_i^{s_i}} T_{\rho_1,\ldots,\rho_d}(f,x) \right\|_{\mathbf{q}} \\
&= \left\| \frac{\partial^{s_i}}{\partial x_i^{s_i}} f(x) - \frac{\partial^{s_i}}{\partial x_i^{s_i}} T_{\rho_1}(f,x) + \frac{\partial^{s_i}}{\partial x_i^{s_i}} T_{\rho_1}(f,x) \right. \\
&\quad \left. - \frac{\partial^{s_i}}{\partial x_i^{s_i}} T_{\rho_1,\rho_2}(f,x) + \cdots - \frac{\partial^{s_i}}{\partial x_i^{s_i}} T_{\rho_1,\ldots,\rho_d}(f,x) \right\|_{\mathbf{q}} \\
&\leqslant c \sum_{j=1}^{d} \rho_j^{-r'_j(1-\frac{s_i}{r'_i})} \left\| \frac{\partial^{s_i}}{\partial x_i^{s_i}} f \right\|_{b_{x_j \mathbf{q}\theta}^{r'_j(1-\frac{s_i}{r'_i})}(R^d)} \\
&\leqslant c\sigma^{-a(\omega-\frac{s_i}{r_i})} \|f\|_{B^{\mathbf{r}}_{\mathbf{p}\theta}(R^d)}.
\end{aligned} \tag{3.6}$$

By (3.1) and (3.6), and Lemma 3, we have

$$E_\sigma^{\mathrm{L}}(S^{\mathbf{r}}_{\mathbf{p}\theta}B(R^d), L^{\mathbf{s}}_{\mathbf{q}}(R^d)) \leqslant \sup_{f \in S^{\mathbf{r}}_{\mathbf{p}\theta}B(R^d)} \|f(\cdot) - L_\nu(f,\cdot)\|_{L^{\mathbf{s}}_{\mathbf{q}}(R^d)} \ll \sigma^{-a(\omega-\eta)}.$$

To find the lower bound, let $\xi \in \Theta_\sigma$, i.e.,

$$\overline{\mathrm{card}}\,\xi = \liminf_{\alpha\to\infty} \frac{\mathrm{card}(\xi \cap [-\alpha,\alpha]^d)}{(2\alpha)^d} \leqslant \sigma.$$

There exists a cube of the form

$$Q_\alpha(m) := \{x \in R^d : \alpha_i \leqslant x_i \leqslant \alpha_i + m_i^{-1}, i = 1,\ldots,d\},$$

where $\alpha \in R^d$, $m_i := (2\sigma)^{a/r_i}$, $i = 1,\ldots,d$, such that its interior Int Q does not contain any point of ξ, that is Int $Q \cap \xi = \emptyset$. This follows easily from the fact that $|Q| = (2\sigma)^{-1}$. Let the univariate function $\lambda(t)$, $t \in R$, satisfy the following conditions: $\lambda(t) \in C^\infty(R)$, supp $\lambda \subset [0,1]$, $0 \leqslant \lambda(t) \leqslant 1$ for $t \in R$, $\lambda(t) = 1$ for $t \in [\frac{1}{4}, \frac{3}{4}]$, and $\|\lambda^{(s_i)}\|_{p_i} \geqslant C > 0$, $i = 1,2,\ldots,d$. Set

$$f_m(x) = \beta \prod_{j=1}^{d} \lambda(m_j(x_j - \alpha_j)),$$

where β is a positive constant which will be determined later. It is easy to see that $f_m(x) \in C^\infty(R^d)$, supp $f_m \subset Q_\alpha(m)$, $I_\xi f_m = 0$, and

$$\|f_m(x)\|_{\mathbf{p}} = \beta \prod_{i=1}^{d} \left(m_i^{-\frac{1}{p_i}} \|\lambda\|_{L_{p_i}[0,1]} \right) \leqslant c_{i_1} \beta \prod_{i=1}^{d} m_i^{-\frac{1}{p_i}}. \tag{3.7}$$

For $1 \leqslant i \leqslant d$, by the Minkowski integral inequality, we have

$$\begin{aligned}
\|\Delta_{t_i}^{k_i} f_m(\cdot)\|_{\mathbf{p}} &= \bigg\| \int_0^{t_i} du_1 \cdots \int_0^{t_i} \frac{\partial^{k_i}}{\partial x_i^{k_i}} f_m(x_1, \ldots, x_i + u_1 \\
&\quad + \cdots + u_{k_i}, x_{i+1}, \ldots, x_d)\, du_{k_i} \bigg\|_{\mathbf{p}} \\
&= \bigg\| \int_0^{t_i} du_1 \cdots \int_0^{t_i} \beta m_i^{k_i} \lambda^{(k_i)}(m_i(x_i + u_1 \\
&\quad + \cdots + u_{k_i} - \alpha_i)) \prod_{\substack{j=1 \\ j \neq i}}^{d} \lambda(m_j(x_j - \alpha_j))\, du_{k_i} \bigg\|_{\mathbf{p}} \\
&\leqslant \beta m_i^{k_i} |t_i|^{k_i} \prod_{j=1}^{d} m_j^{-1/p_j} \|\lambda^{(k_i)}\|_{L_{p_i}[0,1]} \prod_{\substack{j=1 \\ j \neq i}}^{d} \|\lambda\|_{L_{p_j}[0,1]} \\
&\leqslant c_{i_2} \beta m_i^{k_i} |t_i|^{k_i} \prod_{j=1}^{d} m_j^{-1/p_j}.
\end{aligned} \tag{3.8}$$

Moreover

$$\|\Delta_{t_i}^{k_i} f_m(\cdot)\|_{\mathbf{p}} \leqslant c'_{i_3} \|f_m\|_{\mathbf{p}} \leqslant c_{i_3} \beta \prod_{i=1}^{d} m_i^{-\frac{1}{p_i}}. \tag{3.9}$$

Thus, by (3.8) and (3.9), we have

$$\|\Delta_{t_i}^{k_i} f_m(\cdot)\|_{\mathbf{p}} \leqslant c_{i_4} \beta \min\{1, (m_i |t_i|)^{k_i}\} \prod_{j=1}^{d} m_j^{-\frac{1}{p_j}}. \tag{3.10}$$

Hence,

$$\begin{aligned}
\|f_m\|_{b^{r_i}_{x_i \mathbf{p} \theta}(R^d)} &= \left(\int_R \left(\frac{\|\Delta_{t_j}^{k_j} f(\cdot)\|_{\mathbf{p}}}{|t_j|^{r_j}} \right)^{\theta} \frac{dt_j}{|t_j|} \right)^{\frac{1}{\theta}} \\
&\leqslant c_{i_5} \beta \prod_{j=1}^{d} m_j^{-\frac{1}{p_j}} \left(\int_0^{m_i^{-1}} m_i^{k_i \theta} y^{(k_i - r_i)\theta - 1}\, dy + \int_{m_i^{-1}}^{\infty} y^{-r_i \theta - 1}\, dy \right)^{\frac{1}{\theta}} \\
&= c_i \beta m_i^{r_i} \prod_{j=1}^{d} m_j^{-\frac{1}{p_j}} = c_i \beta (2\sigma)^a \prod_{j=1}^{d} m_j^{-\frac{1}{p_j}}.
\end{aligned} \tag{3.11}$$

The estimate (3.11) holds also in the case $\theta = \infty$. By (3.7) and (3.11), if we let $\beta = c\prod_{j=1}^{d} m_j^{\frac{1}{p_j}}(2\sigma)^{-a}$, $c = (c_i + c_{i_1})^{-1}$, then $f_m \in S^{\mathbf{r}}_{\mathbf{p}\theta}B(R^d)$. Moreover, for $1 \leqslant i \leqslant d$,

$$\left\|\frac{\partial^{s_i} f_m(x)}{\partial x_i^{s_i}}\right\|_{\mathbf{q}} = \beta m_i^{s_i-\frac{1}{q_i}}\|\lambda^{(s_i)}\|_{L_{q_i}[0,1]}\prod_{\substack{j=1\\ j\neq i}}^{d}\left(m_j^{-\frac{1}{q_j}}\|\lambda\|_{L_{q_j}[0,1]}\right)$$

$$\geqslant c\beta m_i^{s_i-\frac{1}{q_i}}\prod_{\substack{j=1\\ j\neq i}}^{d}\left(m_j^{-\frac{1}{q_j}}\|\lambda\|_{L_{q_j}[\frac{1}{4},\frac{3}{4}]}\right)$$

$$\geqslant c'\beta m_i^{s_i}\prod_{j=1}^{d} m_j^{-\frac{1}{q_j}}. \tag{3.12}$$

For any $\xi \in \Theta_\sigma$, by (3.7) and (3.12), we have

$$d(I_\xi^{-1}(I_\xi f_m) \cap S^{\mathbf{r}}_{\mathbf{p}\theta}B(R^d), L^{\mathbf{s}}_{\mathbf{q}}(R^d))$$

$$\geqslant \|f_m\|_{L^{\mathbf{s}}_{\mathbf{q}}(R^d)} = \|f_m\|_{\mathbf{q}} + \sum_{i=1}^{d}\left\|\frac{\partial^{s_i} f}{\partial x_i^{s_i}}\right\|_{\mathbf{q}}$$

$$\geqslant c_1\sigma^{-a}\prod_{i=1}^{d} m_i^{\frac{1}{p_i}-\frac{1}{q_i}} + c'\sum_{i=1}^{d} m_i^{s_i}\sigma^{-a}\prod_{j=1}^{d} m_j^{\frac{1}{p_j}-\frac{1}{q_j}}$$

$$\geqslant c_2\sigma^{-a(1-\sum_{i=1}^{d}\frac{1}{r_i}(\frac{1}{p_i}-\frac{1}{q_i})-\eta)} = c_2\sigma^{-a(\omega-\eta)}. \tag{3.13}$$

By the definition of $\Delta_\sigma(S^{\mathbf{r}}_{\mathbf{p}\theta}B(R^d), L^{\mathbf{s}}_{\mathbf{q}}(R^d))$, we get

$$\Delta_\sigma(S^{\mathbf{r}}_{\mathbf{p}\theta}B(R^d), L^{\mathbf{s}}_{\mathbf{q}}(R^d)) \gg \sigma^{-a(\omega-\eta)}.$$

Then an application of (1.2) completes the proof of Theorem 1. □

4. Proof of Theorems 2 and 3

Proof of Theorem 2. Let $p = p_1 = p_2 = \cdots = p_d$, $q = q_1 = \cdots = q_d$, by (1.1), (1.3), (3.1), (3.6), and Lemma 3, we can get the upper bound.

To find the lower bound, let $\xi \in \Theta_\sigma$, i.e.,

$$\overline{\mathrm{card}}\,\xi = \liminf_{\alpha\to\infty}\frac{\mathrm{card}(\xi \cap [-\alpha,\alpha]^d)}{(2\alpha)^d} \leqslant \sigma,$$

$v \in Z^d$, $m_i := (2\sigma)^{a/r_i}$, $i = 1,\ldots,d$,

$$Q_v(m) := \{x \in R^d : v_i m_i^{-1} \leqslant x_i < (v_i+1)m_i^{-1}, i = 1,\ldots,d\},$$

$$\mathscr{Z}_{\xi,\alpha} := \{v \in Z^d : \xi \cap [-\alpha,\alpha]^d \cap \mathrm{Int}\,Q_v(m) = \emptyset, \mathrm{Int}\,Q_v(m) \cap [-\alpha,\alpha]^d \neq \emptyset\},$$

where the set $\mathrm{Int}\, Q_v(m)$ denotes the interior of the cube $Q_v(m)$ for any $v \in Z^d$. Then we see that the natural number

$$n := \mathrm{card}(\mathscr{Z}_{\xi,\alpha}) \geqslant \prod_{i=1}^{d}(2[\alpha m_i]) - \mathrm{card}(\xi \cap [-\alpha, \alpha]^d),$$

and

$$\lim_{\alpha\to\infty} \frac{\mathrm{card}(\mathscr{Z}_{\xi,\alpha})}{(2\alpha)^d} \geqslant \prod_{i=1}^{d} m_i - \sigma = \sigma. \tag{4.1}$$

Let

$$f_m(x) := \sum_{v\in\mathscr{Z}_{\xi,\alpha}} b_v \prod_{j=1}^{d} \phi(m_j x_j - v_j), \quad b_v \in R,$$

where the non-zero function $\phi(\cdot) \in C^\infty(R)$ and $\mathrm{supp}\,\phi(t) \subset [0,1]$, $\|\phi^{(s_i)}\|_q \geqslant c > 0$, $i = 1, \ldots, d$. Then it is easy to verify that $I_\xi(f_m) = 0$,

$$\|f_m(\cdot)\|_q = \left(\prod_{i=1}^{d} m_i^{-1}\right)^{\frac{1}{q}} \|b\|_{l_q^n} \|\phi\|_{L_q[0,1]}^d \quad (b = \{b_v\}_{v\in\mathscr{Z}_{\xi,\alpha}} \in R^n), \tag{4.2}$$

$$\left\|\frac{\partial^{s_i} f_m(\cdot)}{\partial x_i^{s_i}}\right\|_q = \left(\prod_{j=1}^{d} m_j^{-1}\right)^{\frac{1}{q}} m_i^{s_i} \|b\|_{l_q^n} \|\phi\|_{L_q[0,1]}^{d-1} \|\phi^{(s_i)}\|_{L_q[0,1]} \quad (i = 1, \ldots, d), \tag{4.3}$$

$$\|f_m(\cdot)\|_{pq} \leqslant (2[\alpha+1]+2)^{d(1/q-1/p)} \|f_m(\cdot)\|_p, \tag{4.4}$$

$$\|\Delta_{t_i}^{k_i} f_m(\cdot)\|_{pq} \leqslant (2[\alpha+1] + 2[k_i|t_i|] + 4)^{d(1/q-1/p)} \|\Delta_{t_i}^{k_i} f_m(\cdot)\|_p. \tag{4.5}$$

By the Minkowski integral inequality, we have

$$\begin{aligned} &\|\Delta_{t_i}^{k_i} f_m(\cdot)\|_p \\ &\quad = \left\| \int_0^{t_i} du_1 \cdots \int_0^{t_i} \frac{\partial^{k_i}}{\partial x_i^{k_i}} f_m(x_1, \ldots, x_i + u_1 + \cdots + u_{k_i}, x_{i+1}, \ldots, x_d)\, du_{k_i} \right\|_p \\ &\quad \leqslant c_i' \left(\prod_{j=1}^{d} m_j^{-1}\right)^{\frac{1}{p}} m_i^{k_i} |t_i|^{k_i} \|b\|_{l_p^n}. \end{aligned} \tag{4.6}$$

Moreover, by (4.2) and (4.4),

$$\|\Delta_{t_i}^{k_i} f_m(\cdot)\|_{pq} \leqslant c_i' \|f_m\|_{pq} \leqslant c_i'' (2[\alpha+1]+2)^{d(1/q-1/p)} \left(\prod_{i=1}^{d} m_i^{-1}\right)^{\frac{1}{p}} \|b\|_{l_p^n}. \tag{4.7}$$

By (4.5), (4.6), and (4.7), we have

$$\begin{aligned}
\|f_m\|_{b^{r_i}_{x_i pq\theta}(R^d)} &= \left(\int_R \left(\frac{\|\Delta^{k_i}_{t_i} f_m(\cdot)\|_{pq}}{|t_i|^{r_i}}\right)^\theta \frac{dt_i}{|t_i|}\right)^{\frac{1}{\theta}} \\
&= \left(\int_{|t_i|\leqslant m_i^{-1}} + \int_{|t_i|\geqslant m_i^{-1}} \left(\frac{\|\Delta^{k_i}_{t_i} f_m(\cdot)\|_{pq}}{|t_i|^{r_i}}\right)^\theta \frac{dt_i}{|t_i|}\right)^{\frac{1}{\theta}} \\
&\leqslant c_i(2[\alpha+1]+2[k_i m_i^{-1}]+4)^{d(1/q-1/p)} \left(\prod_{j=1}^d m_j^{-1}\right)^{\frac{1}{p}} \|b\|_{l_p^n} \\
&\quad \times \left(\int_0^{m_i^{-1}} m_i^{k_i\theta} y^{(k_i-r_i)\theta-1}\,dy + \int_{m_i^{-1}}^\infty y^{-r_i\theta-1}\,dy\right)^{\frac{1}{\theta}} \\
&= c_i(2[\alpha+1]+2[k_i m_i^{-1}]+4)^{d(1/q-1/p)} \left(\prod_{j=1}^d m_j^{-1}\right)^{\frac{1}{p}} \|b\|_{l_p^n} m_i^{r_i}. \qquad (4.8)
\end{aligned}$$

By (4.4) and (4.8),

$$\|f_m(\cdot)\|_{B^{\mathbf{r}}_{pq\theta}(R^d)} \leqslant c(2[\alpha+1]+c'+4)^{d(1/q-1/p)} \left(\prod_{i=1}^d m_i^{-1}\right)^{\frac{1}{p}} \sigma^a \|b\|_{l_p^n},$$

$$c' = \max\{2[k_i], i=1,\ldots,d\}. \qquad (4.9)$$

By (4.2), (4.3), and (4.9), we have

$$\begin{aligned}
E_\xi &:= \sup\{\|f\|_{L^{\mathbf{s}}_q(R^d)} : \ I_\xi(f)=0,\ f\in S^{\mathbf{r}}_{pq\theta}B(R^d)\} \\
&\geqslant \sup\left\{\frac{\|f_m(\cdot)\|_{L^{\mathbf{s}}_q(R^d)}}{\|f_m(\cdot)\|_{B^{\mathbf{r}}_{pq\theta}(R^d)}} : \ b\in R^n,\ b\neq 0\right\} \\
&\geqslant \sup\left\{\frac{(\prod_{i=1}^d m_i^{-1})^{\frac{1}{q}}\|b\|_{l_q^n}\left(\|\phi\|^d_{L_q[0,1]} + \sum_{i=1}^d m_i^{s_i}\|\phi\|^{d-1}_{L_q[0,1]}\|\phi^{(s_i)}\|_{L_q[0,1]}\right)}{c(2[\alpha+1]+c'+4)^{d(1/q-1/p)}(\prod_{i=1}^d m_i^{-1})^{\frac{1}{p}}\sigma^a\|b\|_{l_p^n}} \right. \\
&\qquad \left. : \ b\in R^n, b\neq 0\right\} \\
&= c_1\left(\prod_{i=1}^d m_i^{-1}\right)^{1/q-1/p} \sigma^{-a(1-\eta)} \frac{\sup\left\{\frac{\|b\|_{l_q^n}}{\|b\|_{l_p^n}} : \ b\in R^n, b\neq 0\right\}}{(2[\alpha+1]+c'+4)^{d(1/q-1/p)}}. \qquad (4.10)
\end{aligned}$$

For $1 < q \leqslant p < \infty$, notice that

$$\sup\left\{\frac{\|b\|_{l_q^n}}{\|b\|_{l_p^n}} : b \in R^n, b \neq 0\right\} \geqslant n^{1/q-1/p} = (\mathrm{card}(\mathscr{Z}_{\xi,\alpha}))^{(1/q-1/p)}, \tag{4.11}$$

by (4.1),

$$\begin{aligned}\lim_{\alpha\to\infty} & \frac{\mathrm{card}(\mathscr{Z}_{\xi,\alpha})}{(2[\alpha+1]+c'+4)^d} \\ &= \lim_{\alpha\to\infty}\frac{\mathrm{card}(\mathscr{Z}_{\xi,\alpha})}{(2\alpha)^d}\frac{(2\alpha)^d}{(2[\alpha+1]+c'+4)^d} \geqslant \sigma.\end{aligned} \tag{4.12}$$

By (4.10)–(4.12), we have

$$E_\xi \gg \sigma^{-(1/q-1/p)-a(1-\eta)+(1/q-1/p)} = \sigma^{-a(1-\eta)}.$$

This is

$$\tfrac{1}{2}\Delta_\sigma(S^{\mathbf{r}}_{pq\theta}B(R^d), L^{\mathbf{s}}_q(R^d)) \gg \sigma^{-a(1-\eta)}.$$

By (1.2), we complete the proof of Theorem 2. $\square$

Proof of Theorem 3. Upper estimate. Let $p = p_1 = \cdots = p_d$, $q = q_1 = \cdots = q_d$, by (1.1), (1.3), (3.1), (3.6), and Lemma 3, we have

$$\begin{aligned}E^{\mathrm{L}}_\sigma(S^{\mathbf{r}}_{p\theta}B(R^d), L^{\mathbf{s}}_{qp}(R^d)) &\leqslant \sup_{f\in S^{\mathbf{r}}_{p\theta}B(R^d)}\|f(\cdot)-L_\nu(f,\cdot)\|_{L^{\mathbf{s}}_{qp}(R^d)} \\ &\leqslant \sup_{f\in S^{\mathbf{r}}_{p\theta}B(R^d)}\|f(\cdot)-L_\nu(f,\cdot)\|_{L^{\mathbf{s}}_{p}(R^d)} \ll \sigma^{-a(1-\eta)}.\end{aligned}$$

Lower estimate. Notice that, for $1 < q \leqslant p < \infty$,

$$\|f_m\|_{qp} \geqslant (2[\alpha+1]+2)^{d(1/p-1/q)}\|f_m\|_q,$$

by (4.2), (4.3), (4.9), and (4.11), we have

$$\begin{aligned}E'_\xi &:= \sup\{\|f\|_{L^{\mathbf{s}}_{qp}(R^d)} : I_\xi(f)=0,\ f\in S^{\mathbf{r}}_{p\theta}B(R^d)\} \\ &\geqslant \sup\left\{\frac{\|f_m(\cdot)\|_{L^{\mathbf{s}}_{qp}(R^d)}}{\|f_m(\cdot)\|_{B^{\mathbf{r}}_{p\theta}(R^d)}} : b\in\mathbf{R}^n,\ b\neq 0\right\} \\ &\geqslant \sup\left\{\frac{(2[\alpha]+2)^{d(1/p-1/q)}(\prod_{i=1}^d m_i^{-1})^{\frac{1}{q}}\|b\|_{l_q^n}\left(\|\phi\|^d_{L_q[0,1]}+\sum_{i=1}^d m_i^{s_i}\|\phi\|^{d-1}_{L_q[0,1]}\|\phi^{(s_i)}\|_{L_q[0,1]}\right)}{c(\prod_{i=1}^d m_i^{-1})^{\frac{1}{p}}\sigma^a\|b\|_{l_p^n}}\right. \\ &\qquad\left. : b\in R^n, b\neq 0\right\} \\ &= c'_1\left(\prod_{i=1}^d m_i^{-1}\right)^{1/q-1/p}\sigma^{-a(1-\eta)}\frac{\sup\left\{\frac{\|b\|_{l_q^n}}{\|b\|_{l_p^n}} : b\in R^n, b\neq 0\right\}}{(2[\alpha+1]+2)^{d(1/q-1/p)}} \\ &\gg \sigma^{-a(1-\eta)}.\end{aligned}$$

This is

$$\frac{1}{2}\Delta_\sigma(S^{\mathbf{r}}_{p\theta}B(R^d), L^{\mathbf{s}}_{qp}(R^d)) \gg \sigma^{-a(1-\eta)}.$$

By (1.2), we complete the proof of Theorem 3. □

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References

[1] A. Benedec, R. Panzone, The spaces L^p with mixed norm, Duke Math. J. 28 (3) (1961) 301–324.
[2] G.S. Fang, Whittaker–Kotelnikov–Shannon sampling theorem and aliasing error, J. Approx. Theory 85 (1996) 115–131.
[3] G.S. Fang, F.J. Hickernell, H. Li, Approximation on anisotropic Besov classes with mixed norms by standard information, J. Complexity 21 (2005) 294–313.
[4] J.J.F. Fourier, J. Stewart, Amalgams of L^p and l^q, Bull. (New Series) of the Am. Math. Soc. 13 (1) (1985) 1–22.
[5] Y.J. Jiang, Optimal recovery of anisotropic Besov classes and sampling theorem in vector norm, East J. Approx. 8 (2) (2002) 279–290.
[6] Y.J. Jiang, Y.P. Liu, Average Widths and optimal recovery of multivariate Besov classes in $L_p(R^d)$, J. Approx. Theory 102 (2000) 155–170.
[7] G.G. Magaril-Il'jaev, Inequalities of Bernstein–Nikolskii type and approximation of generalized Sobolev classes, Proc. Steklov Inst. Math. 173 (4) (1987) 205–220.
[8] C.A. Michelli, T.J. Rivlin, Lecture on optimal recovery, Lecture Notes in Mathematics, vol. 1129, Springer, Berlin, New York, 1985.
[9] S.M. Nikol'skii, Approximation of Functions of Several Variables and Imbedding Theorems, Springer, Berlin, Heidelberg, 1975.
[10] H.-J. Schmeisser, H. Triebel, Topics in Fourier Analysis and Function Spaces, Wiley, Chichester, 1987.
[11] Y.S. Sun, Optimal recovery on a differentiable function class (in Chinese), KEXUE TONGBAO 31 (11) (1986) 809–812.
[12] V.N. Temlyakov, Approximate recovery of periodic functions of several variables, Mat Sbornik 128 (1985) 256–268 (English translation in Math. USSR Sbornik 56 (1987) 249–261).
[13] V.N. Temlyakov, On approximate recovery of functions with bounded mixed derivative, J. Complexity 9 (1993) 41–59.
[14] J.F. Traub, H.A. Woźniakowski, A General Theory of Optimal Algorithms, Academic Press, New York, 1980.
[15] J.J. Wang, G.S. Fang, Multidimensional sampling theorem and estimate of aliasing error, Acta Math. Appl. Sin. 19 (4) (1996) 481–488 (in Chinese).
[16] H.P. Wang, Y.S. Sun, Kolmogorov widths between the anisotropic space and the space of functions with mixed smoothness, J. Approx. Theory 126 (2004) 52–59.
[17] G.Q. Xu, Y.S. Sun, Y.P. Liu, N-widths and average widths of Besov classes in Sobolev spaces, Acta Math. Sin., Engl. Ser. 22 (6) (2006) 1667–1678.